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AREAL SPACES BASED ON GEOMETRICAL THEORY OF PARAMETER-INVARIANT MULTIPLE INTEGRALS

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ABSTRACT

Areal spaces of different types have been studied by several authors, including Cartan [1], Davies [2], Kawaguchi [3] and several others. The study of areal spaces through parameter invariant multiple integrals is of great significance as the geometrical theory of areal spaces based on multiple integrals by Rund [8, 9] generalises both Finsler as well as Cartan spaces. Following Rund's approach, various aspects of areal spaces have been studied by Rastogi [5, 6, 7]. The aim of this paper is to continue that study and give some fundamental relations based on geometrical theory of multiple integrals. In this paper, I have defined and studied torse forming vector fields as well as defined and studied concurrent vector fields in an areal space X4, parameterised by a subspace C2. Besides fundamental tensors and relations, I have also studied special areal spaces like C-reducible areal spaces and some of their properties.

KEYWORDS: *Areal Spaces, Torse Forming Vector Fields. Concurrent Vector Fields, C-Reducible Areal Spaces*

INTRODUCTION

Let Xn be an n-dimensional differentiable manifold with local coordinates xi (i = 1,..., n) and Cr be an r-dimensional subspace represented by

$$x^i = x^i(t^A),\ A = 1, \ldots, r,\ (r< n) \tag{1}$$

where tA denotes a system of r independent parameters on Cr such that Rund [8] x'iA = ∂ xi/ ∂tA. In such a space let us suppose that we are given a function L (xh, x'hE) of n + nr, variables xh, x'hE, which satisfies fundamental conditions given in Rund [9] such that

$$(\partial'^A_i L)\, x'^i_B = L\, \delta^A_B \tag{2}$$

For such a manifold the metric tensor can be expressed as [9]

$$g^{A\,B}_{i\ j}(x^h, x'^h_E) = (1/2)\, r\, (\partial'^B_j\, \partial'^A_i\, L^{2/r}) \tag{3}$$

If $g^{i\ j}_{A\,B}$ is the reciprocal tensor of the metric tensor $g^{A\,B}_{i}$, we have

$$g^{A\,C}_{i\ k}\, g^{i\ j}_{A\,B} = \delta^j_k\, \delta^C_B \tag{4}$$

On the basis of this metric tensor we have [9]

$$C^{A\,B\,D}_{i\ j\ k} = (1/2)\, (\partial'^D_k\, g^{A\,B}_{i\ j}) \tag{5}$$

and for $\partial i = \partial/\partial xi$

$$V^{(A\,B)}_{i\ j\ ,k} = (1/2)(\partial_i g^{A\,B}_{k\ j} + \partial_j g^{A\,B}_{i\ k} - \partial_k g^{A\,B}_{i\ j}), \tag{6}$$

which for

$$f^m_{EF} = \partial_F x'^m_E,\ G^{k\ h}_{B\ D} = (1/2)\,(g^{k\ h}_{B\ D} + g^{k\ h}_{D\ B})\ ,\ C^{A\,B\,E\,F}_{i\ j\ k\ m} = \partial'^F_m C^{A\,B\,E}_{i\ j\ k},$$

and

$$P^{h\ A}_{ij\ D} = G^{k\ h}_{B\ D}(V^{(A\,B)}_{i\ j\ ,k} - C^{A\,B\,E\,F}_{i\ j\ k\ m} f^m_{EF})\text{defines [9]}$$

$$\Gamma^i_{kj} = r^{-1}(P^{i\ B}_{kj\ B} + x'^h_A \partial'^B_k P^{i\ A}_{hj\ B}) \tag{7}$$

such that $\Gamma^i_{kj}\, x'^k_A = P^{i\ B}_{kj\ A}\, x'^k_B$

For a set of linearly independent differentiable vector field X^i_A, tangent to C_r in X_n, we have following covariant derivative of Xi_A, Rund [9]

$$X^i_{A/j} = \partial_j X^i_A - (\partial'^D_k X^i_A)\, \Gamma^k_{mj}\, x'^m_D + \Gamma^i_{jm} X^m_A.. \tag{8}$$

We also have for $T^j_{kh} = \Gamma^j_{kh} - \Gamma^j_{hk}$,

$$Ç_{(k,h)}\{X^i_{A/kh}\} = X^j_A K*^i_{jkh} - (\partial'^B_m X^i_A)\, K^m_{pkh}\, x'^p_B + T^j_{kh} X^i_{A/j} \tag{9}$$

where

$$K^i_{jkh} = Ç_{(k,h)}\{\partial_h \Gamma^i_{jk} - (\partial'^A_p \Gamma^i_{jk})\, \Gamma^p_{mh}\, x'^m_A + \Gamma^i_{mh} \Gamma^m_{jk}\} \tag{10}$$

and

$$K*^i_{jkh} = K^i_{jkh} + T^i_{mj} T^m_{hk} + Ç_{(k,h)}\{T^i_{kj/h} + T^i_{km} T^m_{hj}\} \tag{11}$$

In an earlier paper [7], I have defined following entities:

$$\partial_h g^{A\,B}_{i\ j} = M^{A\,B}_{i\ j\,h} + M^{B\,A}_{j\ i\,h},\ M^k_{ih} = g^{k\ j}_{A\,B} M^{A\,B}_{i\ j\,h},\ M'^i_{kj} = M^i_{kj} - C^{i\ A}_{k\ m} M^m_{rj}\, x'^r_A \tag{12}$$

and

$$A^{i\ E}_{m\ h} = L^{1/r}\,(r)^{1/2} C^{i\ E}_{m\ h},\ C^{k\,E}_{i\ h} = C^{A\,B\,E}_{i\ j\ h}\, g^{k\ j}_{A\,B} \tag{13}$$

If covariant differential in an areal space X_n is defined by

$$D\,X^I_A = X^i_{A,\,h}\, dx^h + X^i_{A\,,,}{}^E_h\ D\,m^h_E \tag{14}$$

where mhE represents a unit vector in X_n, such that $m^h_E = L^{-1/r}\,(r)^{-1/2}\, x'^h_E$ and

$$X^i_{A\,,\,h} = \partial_h X^i_A - (\partial'^E_k X^i_A)\, M'^k_{rh}\, x'^r_E + X^m_A M'^i_{mh} \tag{15}$$

and

$$X^i_{A\,,,}{}^E_h = L^{1/r}\,(r)^{1/2}\, \partial'^E_h X^i_A + X^m_A A^{i\ E}_{m\ h} \tag{16}$$

Alternatively, for, $d\,X^i_A = (\partial_h X^i_A)\, dx^h + (\partial'^E_h X^i_A)\, dx'^h_E$, $D\,X^i_A$ can also be expressed as

$$D\,X^i_A = d\,X^i_A + C^{i\ E}_{k\ m} X^k_A\, dx'^m_E + M^i_{km} X^k_A\, dx^m \tag{17}$$

In place of definition (16), we can also use

$$T^{i\ B}_{A\ j//}{}^{D}_{k} = \partial'^{D}_{k} T^{i\ B}_{A\ j} + g^{i\ t}_{F E} C^{F\ E D}_{m\ t\ k} - g^{t\ u}_{E\ F} C^{E D F}_{j\ k\ u} T^{i\ B}_{A\ t} \quad (18)$$

which gives $X^{i}_{A,,}{}^{D}_{k} = L^{1/r} (r)^{1/2} X^{i}_{A//}{}^{D}_{k}$, $g^{A B}_{i\ j//}{}^{D}_{k} = 0$.

Corresponding to covariant derivative defined by (1.15), we can obtain

$$X^{i}_{A,hk} - X^{i}_{A,kh} = X^{j}_{A} R^{i}_{jkh} - (\partial'^{B}_{m} X^{i}_{A}) R^{m}_{pkh} x'^{p}_{B} + X^{i}_{A,j} T'^{j}_{kh} \quad (19)$$

where $T'^{j}_{kh} = M'^{j}_{kh} - M'^{j}_{hk}$ and

$$R^{i}_{jkh} = Ç_{(k,h)}\{\partial_{h} M'^{i}_{kj} - (\partial'^{A}_{m} M'^{i}_{kj}) M'^{m}_{ph} x'^{p}_{A} + M'^{i}_{hp} M'^{p}_{kj}\} \quad (20)$$

Space X4 Parameterised by C2

In case of n = 4 and r = 2, i.e., X_4 parameterised by C_2, we shall have

$$L^{A}_{i} = (1/2) L^{-1/2} (\partial'^{A}_{i} L) \quad (21)$$

If $L^{i}_{A} = L^{-1/2} x'^{i}_{A}$, it satisfies $L^{A}_{i} L^{i}_{A} = 1$, i.e., it is reciprocal of $L^{A}_{i.}$

From equation (21), we can obtain

$$L^{1}_{i} x'^{i}_{1} = (1/2) L^{1/2},\ L^{1}_{i} x'^{i}_{2} = 0,\ L^{2}_{i} x'^{i}_{1} = 0,\ L^{2}_{i} x'^{i}_{2} = (1/2) L^{1/2} \quad (22)$$

Or alternatively we have

$$L^{A}_{i} x'^{i}_{B} = (1/2) L^{1/2} \delta^{A}_{B} \quad (23)$$

which gives $L^{A}_{i} x'^{i}_{A} = L^{1/2}$

From equation (21), we can also have $L^{A B}_{i\ j} = \partial'^{B}_{j} L^{A}_{i}$, and

$$L^{A B}_{i\ j} = L^{-1/2}\{(1/2) \partial'^{B}_{j} \partial'^{A}_{i} L - L^{A}_{i} L^{B}_{j}\} \quad (24)$$

which satisfies

$$L^{1 1}_{i\ j} x'^{i}_{1} = - (1/2) L^{1}_{j},\ L^{1 2}_{i\ j} x'^{i}_{1} = (1/2) L^{2}_{j},\ L^{1 1}_{i\ j} x'^{i}_{2} = 0,\ L^{1 2}_{i\ j} x'^{i}_{2} = - L^{1}_{j},$$

$$L^{2 1}_{i\ j} x'^{i}_{1} = - L^{2}_{j},\ L^{2 2}_{i\ j} x'^{i}_{1} = 0,\ L^{2 1}_{i\ j} x'^{i}_{1} = (1/2) L^{1}_{j},\ L^{2 2}_{i\ j} x'^{i}_{2} = - (1/2) L^{2}_{j} \quad (25)$$

Or alternatively we shall have

$$L^{A B}_{i\ j} x'^{i}_{A} = 0,\ L^{A B}_{i\ j} x'^{i}_{B} = - (3/2) L^{A}_{j},\ L^{A B}_{i\ j} x'^{i}_{B} x'^{j}_{A} = - (3/2) L^{1/2} \quad (26)$$

In X_4 parameterised by C_2, from equation (3) we can have

$$g^{A B}_{i\ j} = 2 (L^{1/2} L^{A B}_{i\ j} + L^{A}_{i} L^{B}_{j}) \quad (27)$$

which satisfies

$$g^{A B}_{i\ j} x'^{I}_{A} = 2 L^{1/2} L^{B}_{j},\ g^{A B}_{i\ j} x'^{I}_{B} = - 2 L^{1/2} L^{A}_{j},\ g^{A B}_{i\ j} x'^{i}_{A} x'^{J}_{B} = 2 L \quad (28)$$

In X_4, we can also define

$$h^{A B}_{i\ j} = 2 L^{1/2}(\partial'^{B}_{j} \partial'^{A}_{i} L^{1/2}) = 2 L^{1/2} L^{A B}_{i\ j} = g^{A B}_{i\ j} - 2 L^{A}_{i} L^{B}_{j} \quad (29)$$

which satisfies satisfies $h^{A\,B}_{i\;j}\, x'^{i}_{A} = 0$.

Mutually Orthogonal Unit Vectors In X_4 Parameterised by C_2. Corresponding to LiA, in X_4, we can also define mutually orthogonal unit vectors M^A_i, $N^A_{(1)I}$ and $N^A_{(2)I}$, such that

$$L^A_i\, M^i_A = 0,\; M^A_i\, M^i_A = 1,\; L^A_i\, N^i_{(1)A} = 0,\; M^A_i\, N^i_{(1)A} = 0,\; N^i_{A(1)}\, N^A_{(1)I} = 1,$$
$$L^A_i\, N^i_{(2)A} = 0,\; M^A_i\, N^i_{(2)A} = 0,\; N^A_{(1)I}\, N^i_{(2)A} = 0,\; N^A_{(2)I}\, N^I_{(2)A} = 1 \qquad (30)$$

In X_4, $g^{A\,B}_{i\;j}$ and $h^{A\,B}_{i\;j}$ can explicitly be defined as

$$g^{A\,B}_{i\;j} = 2(L^A_i\, L^B_j + M^A_i\, M^B_j + N^A_{(1)I}\, N^B_{(1)j} + N^A_{(2)I}\, N^B_{(2)j}) \qquad (31)$$

and

$$h^{A\,B}_{i\;j} = 2(M^A_i\, M^B_j + N^A_{(1)I}\, N^B_{(1)j} + N^A_{(2)I}\, N^B_{(2)j}) \qquad (32)$$

Corresponding to these tensors giAjB can be written as

$$g^{i\;j}_{A\,B} = L^i_A\, L^j_B + M^i_A\, M^j_B + N^i_{(1)A}\, N^j_{(1)B} + N^i_{(2)A}\, N^j_{(2)B} \qquad (33)$$

From equation (2.8), we can obtain by virtue of $L^{A\,B\,D}_{i\;j\;k} = \partial'^{D}_{k}\, L^{A\,B}_{i\;j}$,

$$\partial'^{D}_{k}\, h^{A\,B}_{i\;j} = L^{-1/2}\, L^{A\,B}_{i\;j}\, (\partial'^{D}_{k}\, L) + 2\, L^{1/2}\, L^{A\,B\,D}_{i\;j\;k} \qquad (34)$$

From equation (5) by virtue of (29) and (35), we get

$$C^{A\,B\,D}_{i\;j\;k} = L^{1/2}\, L^{A\,B\,D}_{i\;j\;k} + L^A_i\, L^{B\,D}_{j\;k} + L^B_j\, L^{D\,A}_{k\;i} + L^D_k\, L^{A\,B}_{i\;j} \qquad (35)$$

which satisfies

$$(L^{1/2}\, L^{A\,B\,D}_{i\;j\;k} - C^{A\,B\,D}_{i\;j\;k})\, L^i_A\, L^j_B = 0 \qquad (36)$$

If we assume that that $h^{A\,B}_{i\;j} = 0$, equation (2.8) and (3.2) give $g^{A\,B}_{i\;j} = 2\, L^A_i\, L^B_j$, $L^{A\,B}_{i\;j} = 0$, therefore equation (3.6) gives $C^{A\,B\,D}_{i\;j\;k} = 0$. Conversely, if $C^{A\,B\,D}_{i\;j\;k} = 0$, we get

$$\partial'^{D}_{k}\, h^{A\,B}_{i\;j} = -\,2(L^A_i\, L^{B\,D}_{j\;k} + L^B_j\, L^{D\,A}_{k\;i}) \qquad (37)$$

which implies $\partial'^{D}_{k}\, (h^{A\,B}_{i\;j})\, x'^{k}_{D} = 0$. Hence:

Theorem 3.1.: In an areal space X4 parameterised by C_2, vanishing of $h^{A\,B}_{i\;j}$ is the sufficient condition for $C^{A\,B\,D}_{i\;j\;k}$ to vanish. Conversely, if $C^{A\,B\,D}_{i\;j\;k} = 0$, the tensor $h^{A\,B}_{i\;j}$ is homogeneous of degree zero in x'^{k}_{D}.

Tensor $C^{A\,B\,D}_{i\;j\;k}$ In X_4 Parameterised by C_2.

Now we are interested in writing the value of $C^{A\,B\,D}_{i\;j\;k}$, explicitly in terms of unit vectors. Let us assume that tensor $C^{A\,B\,D}_{i\;j\;k}$ is expressed as

$$C^{A\,B\,D}_{i\;j\;k} = C_{(1)}\, M^A_i\, M^B_j\, M^D_k + C_{(2)}\, N^A_{(1)I}\, N^B_{(1)j}\, N^D_{(1)k} + C_{(3)}\, N^A_{(2)I}\, N^B_{(2)j}\, N^D_{(2)k} + \sum(^{A\,B\,D}_{i\;j\;k})\, \{C_{(4)}\, M^A_i\, M^B_j\, N^D_{(1)k} + C_{(5)}\, M^A_i\, M^B_j\, N^D_{(2)k} + C_{(6)}\, N^A_{(1)I}\, N^B_{(1)j}\, M^D + C_{(7)}\, N^A_{(1)I}\, N^B_{(1)j}\, N^D_{(2)k} + C_{(8)}\, N^A_{(2)I}\, N^B_{(2)j}\, M^D_k + C_{(9)}\, N^A_{(2)I}\, N^B_{(2)j}\, N^D_{(1)k} + C_{(10)}\, M^A_i (N^B_{(1)j}\, N^D_{(2)k} + N^B_{(2)j}\, N^D_{(1)k})\} \qquad (38)$$

Multiplying equation (38) by $g^{J\;k}_{B\,D}$, we get by virtue of equation (30) and (33)

$$C^A_i = (C_{(1)} + C_{(6)} + C_{(8)})\, M^A_i + (C_{(2)} + C_{(4)} + C_{(9)}) N^A_{(1)I} + (C_{(3)} + C_{(5)} + C_{(7)})\, N^A_{(2)I},$$

which by virtue of of $C^A_i = C\, M^A_i$ leads to

$$C_{(1)} + C_{(6)} + C_{(8)} = C,\ C_{(2)} + C_{(4)} + C_{(9)} = 0,\ C_{(3)} + C_{(5)} + C_{(7)} = 0 \quad (39)$$

Hence:

Theorem 4.1: In a four-dimensional areal space X4, parameterised by a two-dimensional subspace C2, the tensor CAiBjDk is symmetric in the pair of indices (A, i), (B, j) and (D, k) and it is expressed by equation (38) such that its coefficients satisfy equation (39).

Multiplying equation (39) by M^i_A, $N^i_{(1)A}$ and $N^i_{(2)A}$ respectively, we get by virtue of $C^{A\,B\,D}_{i\,j\,k}\, M^i_A = C^{B\,D}_{j\,k}$, $C^{A\,B\,D}_{i\,j\,k}\, N^i_{(1)A} = {}^1C^{B\,D}_{j\,k}$ and $C^{A\,B\,D}_{i\,j\,k}\, N^i_{(2)A} = {}^2C^{B\,D}_{j\,k}$

$$C^{B\,D}_{j\,k} = C_{(1)}\, M^B_j\, M^D_k + C_{(4)}(M^B_j\, N^D_{(1)k} + M^D_k\, N^B_{(1)j}) + C_{(5)}(M^B_j\, N^D_{(2)k} + M^D_k\, N^B_{(2)j}) + C_{(6)}\, N^B_{(1)j}\, N^D_{(1)k} + C_{(8)}\, N^B_{(2)j}\, N^D_{(2)k} + C_{(10)}(N^B_{(1)j}\, N^D_{(2)k} + N^B_{(2)j}\, N^D_{(1)k}), \quad (40)$$

$${}^1C^{B\,D}_{j\,k} = C_{(2)}\, N^B_{(1)j}\, N^D_{(1)k} + C_{(4)}\, M^B_j\, M^D_k + C_{(6)}(N^B_{(1)j}\, M^D_k + N^D_{(1)k}\, M^B_j) + C_{(7)}(N^B_{(1)j}\, N^D_{(2)k} + N^B_{(2)j}\, N^D_{(1)k}) + C_{(9)}\, N^B_{(2)j}\, N^D_{(2)k} + C_{(10)}(M^B_j\, N^D_{(2)k} + M^D_k\ {}_{(2)j}) \quad (41)$$

and

$${}^2C^{B\,D}_{j\,k} = C_{(3)}\, N^B_{(2)j}\, N^D_{(2)k} + C_{(5)}\, M^B_j\, M^D_k + C_{(7)}\, N^B_{(1)j}\, N^D_{(1)k} + C_{(8)}(N^B_{(2)j}\, M^D_k + N^D_{(2)k}\, M^B_j)$$
$$+ C_{(9)}(N^B_{(2)j}\, N^D_{(1)k} + N^D_{(2)k}\, N^B_{(1)j}) + C_{(10)}(M^B_j\, N^D_{(1)k} + M^D_k\, N^B_{(1)j}) \quad (42)$$

From equations (39), (40), (41), (42) we can further obtain

$$C^{A\,B\,D}_{i\,j\,k} = C^{B\,D}_{j\,k}\, M^A_i + {}^1C^{B\,D}_{j\,k}\, N^A_{(1)I} + {}^2C^{B\,D}_{j\,k}\, N^A_{(2)I} \quad (43)$$

which implies:

Theorem 4.2: In an areal space X4, parameterised by subspace C2, the tensor $C^{A\,B\,k}_{i\,j\,D}$ can be decomposed in terms of the vectors M^A_i, $N^A_{(1)I}$ and $N^A_{(2)I}$ and expressed in the form of equation (4.4).

From equations (39), (40), (41), (42) we can further obtain

$$C^{B\,D}_{j\,k}\, M^J_B = C_{(1)}\, M^D_k + C_{(4)}\, N^D_{(1)k} + C_{(5)}\, N^D_{(2)k} \quad (44)$$

$$C^{B\,D}_{j\,k}\, N^J_{(1)B} = C_{(4)}\, M^D_k + C_{(6)}\, N^D_{(1)k} + C_{(10)}\, N^D_{(2)k} = {}^!C^{B\,D}_{j\,k}\, M^j_B \quad (45)$$

$$C^{B\,D}_{j\,k}\, N^j_{(2)B} = C_{(5)}\, M^D_k + C_{(8)}\, N^D_{(2)k} + C_{(10)}\, N^D_{(1)k} = {}^2C^{B\,D}_{j\,k}\, M^j_B \quad (46)$$

$${}^1C^{B\,D}_{j\,k}\, N^J_{(1)B} = C_{(2)}\, N^D_{(1)k} + C_{(6)}\, M^D_k + C_{(7)}\, N^D_{(2)k} \quad (47)$$

$${}^1C^{B\,D}_{j\,k}\, N^j_{(2)B} = C_{(7)}\, N^D_{(1)k} + C_{(9)}\, N^D_{(2)k} + C_{(10)}\, M^D_k = {}^2C^{B\,D}_{j\,k}\, N^j_{(1)B} \quad (48)$$

$${}^2C^{B\,D}_{j\,k}\, N^J_{(2)B} = C_{(3)}\, N^D_{(2)k} + C_{(8)}\, M^D_k + C_{(9)}\, N^D_{(1)k} \quad (49)$$

From equations (5.5) a, b, c, d, e, f, we can obtain the values of coefficients C(1) to C(10) as follows:

$C_{(1)} = C^{B\,D}_{j\,k}\, M^j_B\, M^K_D$, $C_{(2)} = {}^1C^{B\,D}_{j\,k}\, N^j_{(1)B}\, N^k_{(1)D}$, $C_{(3)} = {}^2C^{B\,D}_{j\,k}\, N^j_{(2)B}\, N^k_{(2)D}$, $C_{(4)} = C^{B\,D}_{j\,k}\, M^j_B\, N^k_{(1)D} = C^{B\,D}_{j\,k}\, N^j_{(1)B}\, M^k_D$, $C_{(5)} = C^{B\,D}_{j\,k}\, M^j_B\, N^k_{(2)D} = C^{B\,D}_{j\,k}\, N^j_{(2)B}\, M^k_D$, $C_{(6)} = C^{B\,D}_{j\,k}\, N^j_{(1)B}\, N^k_{(1)D} = {}^1C^{B\,D}_{j\,k}\, N^j_{(1)B}\, M^k_D$, $C_{(7)} = {}^1C^{B\,D}_{j\,k}\, N^j_{(1)B}\, N^k_{(2)D} = {}^1C^{B\,D}_{j\,k}\, N^j_{(2)B}\, N^k_{(1)D}$, $C_{(8)} = C^{B\,D}_{j\,k}\, N^j_{(2)B}\, N^k_{(2)D} = {}^2C^{B\,D}_{j\,k}\, N^J_{(2)B}\, M^k_D$, $C_{(9)} = {}^1C^{B\,D}_{j\,k}\, N^j_{(2)B}\, N^k_{(2)D} = {}^2C^{B\,D}_{j\,k}\, N^J_{(2)B}\, N^k_{(1)D}$, $C_{(10)} = C^{B\,D}_{j\,k}$

$N^{J}_{(1)B}\, N^{k}_{(2)D} = C^{B\,D}_{\ j\ \ k}\, N^{j}_{(2)B}\, N^{k}_{(1)D} = {}^{1}C^{B\,D}_{\ j\ \ k}\, N^{j}_{(2)B}\, M^{k}_{D}$ (50)

Hence:

Theorem 4.3: In an areal space X4, parameterised by subspace C2, the coefficients C(1) to C(10) of equation (40) are given by equation (50).

Also, from equation (43), we can obtain

$$C^{B\,D}_{\ j\ \ k}\, M^{j}_{B} + {}^{1}C^{B\,D}_{\ j\ \ k}\, N^{j}_{(1)B} + {}^{2}C^{B\,D}_{\ j\ \ k}\, N^{j}_{(2)B} = C^{D}_{\ k} \quad (51)$$

Hence:

Theorem 4.4.: In a four-dimensional areal space X_4, parameterised by C_2, the vector $C^{D}_{\ k}$ satisfies equation (51).

C- Reducible Areal Spaces

In this section we are interested in studying C-reducible areal space X4, parameterised by C2, which has earlier been studied by the author [7]. Here in such a C-reducible areal space we shall have

$$C^{A\,B\,D}_{\ i\ \ j\ \ k} = (1/10)\,(h^{A\,B}_{\ i\ \ j}\, C^{D}_{\ k} + h^{B\,D}_{\ j\ \ k}\, C^{A}_{\ i} + h^{D\,A}_{\ k\ \ i}\, C^{B}_{\ j}) \quad (52)$$

Substituting the value of tensor field $h^{A\,B}_{\ i\ \ j}$ and $C^{D}_{\ k}$,, we can write equation (52) as

$$C^{A\,B\,D}_{\ i\ \ j\ \ k} = (C/5)[3\, M^{A}_{\ i}\, M^{B}_{\ j}\, M^{D}_{\ k} + \textstyle\sum(^{A\,B\,D}_{\ i\ \ j\ \ k})\{M^{A}_{\ i}\,(N^{B}_{(1)j}\, N^{D}_{(1)k} + N^{B}_{(2)j}\, N^{D}_{(1)k})\}] \quad (53)$$

Comparing equations (40) and (53), we can establish

$$C_{(1)} = 3C/5,\ C_{(2)} = 0,\ C_{(3)} = 0,\ C_{(4)} = 0,\ C_{(5)} = 0,\ C_{(6)} = C/5,\ C_{(7)} = 0,\ C_{(8)} = C/5,\ C_{(9)} = 0,\ C_{(10)} = 0. \quad (54)$$

Hence:

Theorem 5.1: If a four-dimensional areal space X4, parameterised by C2, is a C-reducible areal space, its coefficients satisfy equation (54).

The tensors $C^{B\,D}_{\ j\ \ k}$, ${}^{1}C^{B\,D}_{\ j\ \ k}$ and ${}^{2}C^{B\,D}_{\ j\ \ k}$ defined in (40), (41), (42), (43) shall be given by

$$C^{B\,D}_{\ j\ \ k} = (C/5)\,(3\, M^{B}_{\ j}\, M^{D}_{\ k} + N^{B}_{(1)j}\, N^{D}_{(1)k} + N^{B}_{(2)j}\, N^{D}_{(2)k}) \quad (55)$$

$${}^{1}C^{B\,D}_{\ j\ \ k} = (C/5)\,(M^{B}_{\ j}\, N^{D}_{(1)k} + M^{D}_{\ k}\, N^{B}_{(1)j}) \quad (56)$$

$${}^{2}C^{B\,D}_{\ j\ \ k} = (C/5)\,(M^{B}_{\ j}\, N^{D}_{(2)k} + M^{D}_{\ k}\, N^{B}_{(2)j}) \quad (57)$$

Torse Forming Vector Fields

Def. 6.1: A vector field $X^{i}_{A}(x)$, in an areal space X_n parameterised by C_r shall be called torse forming vector field, if it satisfies

$$X^{i}_{A\,,\,k} = \lambda_A\, \delta^{i}_{k} + \mu^{D}_{\ k}\, \sigma_D\, X^{i}_{A} \quad (58)$$

where λ_A is a scalar field and $\mu^{D}_{\ k}\, \sigma_D$ is a non-zero vector field.

From equation (6.1), we can obtain

$$X^{i}_{A\,,\,h,\,k} = \delta^{i}_{h}\, \lambda_{A\,,\,k} + (\mu^{D}_{\ h\,,\,k}\, \sigma_D + \mu^{D}_{\ h}\, \sigma_{D\,,\,k})\, X^{i}_{A} + \mu^{D}_{\ h}\, \sigma_D(\lambda_A\, \delta^{i}_{k} + \mu^{E}_{\ k}\, \sigma_E\, X^{i}_{A})$$

Which by virtue of equation (19) gives on Simplification

$$X^j_A R^i_{jkh} - (\partial'^B_m X^i_A) R^m_{pkh} x'^p_B + X^i_{A,j} T'^j_{kh} = Ç_{(h,k)} \{\delta^i_h(\lambda_{A,k} - \lambda_A \mu^D_k \sigma_D) + X^i_A (\mu^D_{h,k} \sigma_D + \mu^D_h \sigma_{D,k})\} \quad (59)$$

which can be expressed as

$$X^j_A R^i_{jkh} - (\partial'^B_m X^i_A) R^m_{pkh} x'^p_B = Ç_{(h,k)} \{\delta^i_h(\lambda_{A,k} - \lambda_A \mu^D_k \sigma_D) + X^i_A (\mu^D_{h,k} \sigma_D + \mu^D_h \sigma_{D,k} - \mu^D_j \sigma_D M'^j_{kh}) - \lambda_A M'^i_{kh}\} \quad (60)$$

Hence:

Theorem 6.1: A torse forming vector field $X^i_A(x)$, in an areal space X_n parameterised by C_r, satisfies equation (60).

Concurrent Vector Fields

Def. 7.1: A vector field $X^i_A(x)$, in an areal space X_n parameterised by C_r, shall be called concurrent vector field if it satisfies equations

$$X^i_{A,j} = \lambda_A \delta^i_j, \quad (61)$$

$$X^i_A C^{E\,B\,D}_{i\,j\,k} = \varphi_A \varphi^E h^{B\,D}_{j\,k}, \quad (62)$$

where φ_A is a scalar in C_r satisfying $\varphi_A \varphi^A = r\,\varphi$.

Let X^i_A in X_4, parameterised by C_2, be expressed as

$$X^i_A = \alpha L^i_A + \beta M^i_A + \gamma N^i_{(1)A} + \Theta N^i_{(2)A} \quad (63)$$

From equations (62) a and (63), we can write

$$X^i_{A,j} = \lambda_A \delta^i_j = \alpha_{,j} L^i_A + \beta_{,j} M^i_A + \gamma_{,j} N^i_{(1)A} + \Theta_{,j} N^i_{(2)A} + \alpha L^i_{A,j} + \beta M^i_{A,j} + \gamma N^i_{(1)A,j} + \Theta N^i_{(2)A,j}$$

which on simplification by virtue of

$$L^i_{A,j} = 0,\ M^i_{A,j} = \beta(h_j N^i_{(1)A} + j_j N^i_{(2)A}),\ N^i_{(1)A,j} = \gamma(-h_j M^i_A + k_j N^i_{(2)A}),\ N^i_{(2)A,j} = \Theta(j_j M^i_A - k_j N^i_{(1)A}) \quad (64)$$

where h_j, j_j and k_j are vector fields similar to, h-vectors of Finsler space, leads to following relations

$$\alpha_{,j} = \lambda_A L^A_j,\ \beta_{,j} = \gamma h_j - \Theta j_j + \lambda_A M^A_j,\ \gamma_{,j} = -\beta h_j + \Theta k_j + \lambda_A N^A_{(1)j},\ \Theta_{,j} = -\beta j_j - \gamma k_j + \lambda_A N^A_{(2)j} \quad (65)$$

From equations (38) and (7.2), we can get

$$X^i_A C^{A\,B\,D}_{i\,j\,k} = \beta\{C_{(1)} M^B_j M^D_k + C_{(4)}(M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j}) + C_{(5)}(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j}) + C_{(6)} N^B_{(1)j} N^D_{(1)k} + C_{(8)} N^B_{(2)j} N^D_{(2)k} + C_{(10)}(N^B_{(1)j} N^D_{(2)k} + N^B_{(2)j} N^D_{(1)k})\} + \gamma\{C_{(2)} N^B_{(1)j} N^D_{(1)k} + C_{(4)} M^B_j M^D_k + C_{(6)}(M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j}) + C_{(7)}(N^B_{(1)j} N^D_{(2)k} + N^D_{(1)k} N^B_{(2)j}) + C_{(9)} N^B_{(2)j} N^D_{(2)k} + C_{(10)}(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j})\} + \Theta\{C_{(3)} N^B_{(2)j} N^D_{(2)k} + C_{(5)} M^B_j M^D_k + C_{(7)} N^B_{(1)j} N^D_{(1)k} + C_{(8)}(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j}) + C_{(9)}(N^B_{(1)j} N^D_{(2)k} + N^D_{(1)k} N^B_{(2)j}) + C_{(10)}(M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j})\} \quad (66)$$

which by virtue of equations (40), (41), (42) can be expressed as

$$X^i_A C^{A\,B\,D}_{i\,j\,k} = \beta C^{B\,D}_{j\,k} + \gamma\, {}^1C^{B\,D}_{j\,k} + \Theta\, {}^2C^{B\,D}_{j\,k} \quad (67)$$

From equations (62) and (67), we can obtain

$$\beta C^{B\,D}_{j\,k} + \gamma\, {}^1C^{B\,D}_{j\,k} + \Theta\, {}^2C^{B\,D}_{j\,k} = 2\,\varphi\, h^{B\,D}_{j\,k} \quad (68)$$

which on simplification and multiplication by M^k_D, $N^k_{(1)D}$ and $N^k_{(2)D}$ respectively shall give

$$C_{(1)}\beta + C_{(4)}\gamma + C_{(5)}\Theta = 4\,\varphi,\ \ C_{(4)}\beta + C_{(6)}\gamma + C_{(10)}\Theta = 0,\ C_{(5)}\beta + C_{(10)}\gamma + C_{(8)}\Theta = 0, \quad (69)$$

$$C_{(6)}\beta + C_{(2)}\gamma + C_{(7)}\Theta = 4\varphi,\ \ C_{(4)}\beta + C_{(6)}\gamma + C_{(10)}\Theta = 0,\ C_{(10)}\beta + C_{(7)}\gamma + C_{(9)}\Theta = 0, \tag{70}$$

$$C_{(8)}\beta + C_{(9)}\gamma + C_{(3)}\Theta = 4\varphi,\ \ C_{(5)}\beta + C_{(10)}\gamma + C_{(8)}\Theta = 0,\ C_{(10)}\beta + C_{(7)}\gamma + C_{(9)}\Theta = 0, \tag{71}$$

Equations (69), (70), (71) help us to give

$$\varphi = \beta C/12,\ C_{(4)}(C_{(9)}C_{(10)} - C_{(7)}C_{(8)}) - C_{(6)}(C_{(5)}C_{(9)} - C_{(8)}C_{(10)}) + C_{(10)}(C_{(5)}C_{(7)} - C_{(10)}^{2}) = 0 \tag{72}$$

Hence:

Theorem 7.1: In a four-dimensional areal space X_4 parameterised by a subspace C_2, a concurrent vector field X^i_A is such that some of its coefficients satisfy equation (72).

In case of a C-reducible areal space, equations (66) can be expressed as

$$X^i_A C^{A\,B\,D}_{i\,j\,k} = \beta(C_{(1)} M^B_j M^D_k + C_{(6)} N^B_{(1)j} N^D_{(1)k} + C_{(8)} N^B_{(2)j} N^D_{(2)k}) + \gamma C_{(6)}(M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j}) + \Theta C_{(8)}(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j}) \tag{73}$$

Using equation (66) in (73) we can obtain equation (72). From equation (72), it is easy to observe that $\gamma = 0$ and $\Theta = 0$. Hence:

Theorem 7.2: If X^I_A is a concurrent vector field in a C-reducible areal space X_4, parameterised by $C_{(2)}$, scalars γ and Θ assume vanishing values.

Corresponding to tensor $C^{A\,B\,D}_{i\,j\,k}$, we have tensor $S^{A\,B\,D\,G}_{i\,j\,k\,m}$ defined as Rastogi [7]

$$S^{A\,B\,D\,G}_{i\,j\,k\,m} = Ç(^{B\,D}_{j\,k})\{g^{p\,t}_{F\,E}\, C^{A\,B\,E}_{i\,j\,t}\, C^{D\,F\,G}_{k\,p\,m}\} \tag{74}$$

Multiplying equation (74) by $X^I_A X^m_G$ and using equation (66) and (67), we can get

$$X^i_A X^m_G S^{A\,B\,D\,G}_{i\,j\,k\,m} = 4\varphi^2 g^{p\,t}_{F\,E}(h^{B\,t}_{j\,E} h^{D\,F}_{k\,p} - h^{D\,E}_{k\,t} h^{B\,F}_{j\,p}) \tag{75}$$

Equation (75), when solved yields

$$X^i_A X^m_G S^{A\,B\,D\,G}_{i\,j\,k\,m} = 0 \tag{76}$$

Hence:

Theorem 7.3: If X^i_A is a concurrent vector field in an areal space X_4 parameterised by C_2, the curvature tensor $S^{A\,B\,D\,G}_{i\,j\,k\,m}$ satisfies equation (76).

Tensor $C^{A\,B\,D}_{i\,j\,k\,,\,r}$ in X_4 parameterised by C_2. Differentiating equation (38) with respect to x^r, and using equation (64), we can obtain after some tedious calculation

$$C^{A\,B\,D}_{i\,j\,k\,,\,r} = A_{(1)r} M^A_i M^B_j M^D_k + A_{(2)r} N^A_{(1)I} N^B_{(1)j} N^D_{(1)k} + A_{(3)r} N^A_{(2)I} N^B_{(2)j} N^D_{(2)k} + \sum(^{A\,B\,D}_{I\,J\,K})\{A_{(4)r} M^A_i M^B_j N^D_{(1)k} + A_{(5)r} M^A_i M^B_j N^D_{(2)k} + A_{(6)r} M^A_i N^B_{(1)j} N^D_{(1)k} + A_{(7)r} N^A_{(1)I} N^B_{(1)j} N^D_{(2)k} + A_{(8)r} M^A_i N^B_{(2)j} N^D_{(2)k} + A_{(9)r} N^A_{(1)I} N^B_{(2)j} N^D_{(2)k} + A_{(10)r} M^A_i(N^B_{(1)j} N^D_{(2)k} + N^D_{(1)k} N^B_{(2)j})\} \tag{77}$$

Where we have put

$$A_{(1)r} = C_{(1)\,,\,r} - 3\gamma C_{(4)} h_r + 3\Theta C_{(5)} j_r,\ A_{(2)r} = C_{(2)\,,\,r} + 3\beta C_{(6)} h_r - 3\Theta C_{(7)} k_r, \tag{78}$$

$$A_{(3)r} = C_{(3)\,,\,r} + 3\beta C_{(8)} j_r + 3\gamma C_{(9)} k_r,\ A_{(4)r} = C_{(4)\,,\,r} + (\beta C_{(1)} - 2\gamma C_{(6)}) h_r + 2\Theta C_{(10)} j_r, \tag{79}$$

$$A_{(5)r} = C_{(5),r} + (\beta\, C_{(1)} + 2\,\Theta\, C_{(8)})\, j_r + (\gamma\, C_{(4)} - \Theta\, C_{(5)})\, k_r - 2\,\gamma\, C_{(10)}\, h_r, \quad (80)$$

$$A_{(6)r} = C_{(6),r} - (\gamma\, C_{(2)} - 2\,\beta\, C_{(4)})\, h_r + \Theta\, C_{(7)}\, j_r - 2\,\Theta\, C_{(10)}\, k_r, \quad (81)$$

$$A_{(7)r} = C_{(7),r} + \beta\, C_{(6)}\, j_r + (\gamma\, C_{(2)} - 2\,\Theta\, C_{(9)})\, k_r + 2\,\beta\, C_{(10)}\, h_r, \quad (82)$$

$$A_{(8)r} = C_{(8),r} + (\Theta\, C_{(3)} + 2\,\beta\, C_{(5)})\, j_r - \gamma\, C_{(9)}\, h_r + 2\,\gamma\, C_{(10)}\, k_r, \quad (83)$$

$$A_{(9)r} = C_{(9),r} - (\Theta\, C_{(3)} - 2\gamma\, C_{(7)})\, k_r + \beta\, C_{(8)}\, h_r + 2\,\beta\, C_{(10)}\, j_r, \quad (84)$$

$$A_{(10)r} = C_{(10),r} + (\beta\, C_{(4)} + \Theta\, C_{(9)})\, j_r + (\beta\, C_{(5)} - \gamma\, C_{(7)})\, h_r + (\gamma\, C_{(6)} - \Theta\, C_{(8)})\, k_r \quad (85)$$

Differentiating equation $X^i_A\, C^{A}_{i}{}^{B}_{j}{}^{k}_{k} = 2\,\varphi\, h^{B}_{j}{}^{D}_{k}$, for X_4 parameterised by C_2, we get on simplification

$$M^B_j\, M^D_k\{\lambda_A(C_{(1)}\, M^A_r + C_{(4)}\, N^A_{(1)r} + C_{(5)}\, N^A_{(2)r}) + \beta\, A_{(1)r} + \gamma\, A_{(4)}r + \Theta\, A_{(5)r}\} + N^B_{(1)j}\, N^D_{(1)k}\{\lambda_A(C_{(2)}\, N^A_{(1)r} + C_{(6)}\, M^A_r + C_{(7)}\, N^A_{(2)r}) + \beta\, A_{(6)r} + \gamma\, A_{(2)r} + \Theta\, A_{(7)r}\} + N^B_{(2)j}\, N^D_{(2)k}\{\lambda_A(C_{(3)}\, N^A_{(2)r} + C_{(8)}\, M^A_r + C_{(9)}\, N^A_{(1)r}) + \beta\, A_{(8)r} + \gamma\, A_{(9)r} + \Theta\, A_{(3)r}\} + (M^B_j\, N^D_{(1)k} + M^D_k\, N^B_{(1)j})\{\lambda_A(C_{(4)}\, M^A_r + C_{(6)}\, N^A_{(1)r} + C_{(10)}\, N^A_{(2)r}) + \beta\, A_{(5)r} + \gamma\, A_{(6)r} + \Theta\, A_{(10)r}\} + (M^B_j\, N^D_{(2)k} + M^D_k\, N^B_{(2)j})\{\lambda_A(C_{(5)}\, M^A_r + C_{(10)}\, N^A_{(1)r} + C_{(8)}\, N^A_{(2)r}) + \beta\, A_{(5)r} + \gamma\, A_{(10)r} + \Theta\, A_{(8)r}\} + (N^B_{(1)j}\, N^D_{(2)k} + N^D_{(1)k}\, N^B_{(2)j})\{\lambda_A(C_{(10)}\, M^A_r + C_{(7)}\, N^A_{(1)r} + C_{(9)}\, N^A_{(2)r}) + \beta\, A_{(10)r} + \gamma\, A_{(7)r} + \Theta\, A_{(9)r}\} = 2\,\varphi_{,r}\, h^{B}_{j}{}^{D}_{k} \quad (86)$$

Multiplying equation (86) by $g^{j}_{B}{}^{k}_{D}$, we get

$$\beta(A_{(1)r} + A_{(6)r} + A_{(8)r}) + \gamma(A_{(2)r} + A_{(4)r} + A_{(9)r}) + \Theta(A_{(3)r} + A_{(5)r} + A_{(7)r}) = 12\,\varphi_{,r} - C\,\lambda_A\, M^A_r \quad (87)$$

Hence:

Theorem 8.1: A four-dimensional areal space X_4 parameterised by C_2 is such that if, X^i_A is a concurrent vector field, coefficients defined by equations (78) to (84) satisfy equation (87).

In case of a C-reducible areal space equations (78) to (85) will change to

$$A_{(1)r} = C_{(1),r},\ A_{(2)r} = 3\,\beta\, C_{(6)}\, h_r,\ A_{(3)r} = 3\,\beta\, C_{(8)}\, j_r,\ A_{(4)r} = \beta\, C_{(1)}\, h_r,\ A_{(5)r} = \beta\, C_{(1)}\, j_r,\ A_{(6)r} = C_{(6),r},$$

$$A_{(7)r} = \beta\, C_{(6)}\, j_r,\ A_{(8)r} = C_{(8),r},\ A_{(9)r} = \beta\, C_{(8)}\, h_r,\ A_{(10)r} = 0. \quad (88)$$

Furthermore, equation (86) shall change to

$$2\,\varphi_{,r}\, h^{B}_{j}{}^{D}_{k} = M^B_j\, M^D_k(\lambda_A\, C_{(1)}\, M^A_r + \beta\, C_{(1),r}) + N^B_{(1)j}\, N^D_{(1)k}\,\{(\lambda_A\, C_{(6)}\, M^A_r) + \beta\, C_{(6),r}\} + N^B_{(2)j}\, N^D_{(2)k}\{\lambda_A(C_{(8)}\, M^A_r) + \beta\, C_{(8),r}\} + (M^B_j\, N^D_{(1)k} + M^D_k\, N^B_{(1)j})\,\lambda_A(C_{(6)}\, N^A_{(1)r}) \quad (8.6)$$

Also, equation (87) can be expressed as

$$\beta\,(C_{(1),r} + C_{(6),r} + C_{(8),r}) = 12\,\varphi_{,r} - C\,\lambda_A\, M^A_r \quad (89)$$

Hence:

Theorem 8.2: A four-dimensional C-reducible areal space X_4 parameterised by C_2, is such that, if X^i_A is a con-current vector field, vectors $C_{(1),r}$, $C_{(6),r}$, $C_{(8),r}$ shall satisfy equation (89).

Tensor $C^{A}_{i}{}^{B}_{j}{}^{D}_{k//m}{}^{E}$ in X_4 Parameterised by C_2.

Using covariant derivative given in equation (18), we can obtain

$$L^{i}_{A//j}{}^{B} = L^{-1}(M^i_A\, M^B_j + N^i_{(1)A}\, N^B_{(1)j} + N^i_{(2)A}\, N^B_{(2)j}) = (1/2)\, L^{-1}\, h^{i}_{A}{}^{j}_{B}, \quad (90)$$

$$M^{i}_{A//j}{}^{B} = L^{-1}(- L^{i}_{A}\, M^{B}_{j} + N^{i}_{(1)A}\, U^{B}_{j} + N^{i}_{(2)A}\, V^{B}_{j}), \tag{91}$$

$$N^{i}_{(1)A//j}{}^{B} = L^{-1}(- L^{i}_{A}\, N^{B}_{(1)j} - M^{i}_{A}\, U^{B}_{j} + N^{i}_{(2)A}\, W^{B}_{j}), \tag{92}$$

$$N^{i}_{(2)A//j}{}^{B} = L^{-1}(- L^{i}_{A}\, N^{B}_{(2)j} - M^{i}_{A}\, V^{B}_{j} - N^{i}_{(1)A}\, W^{B}_{j}), \tag{93}$$

where vector fields U^{B}_{j}, V^{B}_{j} and W^{B}_{j} in an areal space X_4 parameterised by C_2, are similar to V-connection vectors of Finsler space.

Taking this type of covariant derivative of the vector field X^{i}_{A} and using equations (90), (91), (92), (93), we can obtain

$$X^{i}_{A//j}{}^{E} = L^{i}_{A}\{\alpha_{//j}{}^{E} - L^{-1}(\beta\, M^{E}_{j} + \gamma\, N^{E}_{(1)j} + \Theta\, N^{E}_{(2)j})\} + M^{i}_{A}\{\beta_{//j}{}^{E} + L^{-1}(\alpha/2\, M^{E}_{j} - \gamma\, U^{E}_{j} - \Theta\, V^{E}_{j})\} + N^{i}_{(1)A}\{\gamma_{//j}{}^{E} + L^{-1}(\alpha/2\, N^{E}_{(1)j} + \beta\, U^{E}_{j} - \Theta\, W^{E}_{j})\} + N^{i}_{(2)A}\{\Theta_{//j}{}^{E} + L^{-1}(\alpha/2\, N^{E}_{(2)j} + \beta\, V^{E}_{j} + \gamma\, W^{E}_{j})\} \tag{94}$$

Differentiating equation (36) and using equations (90), (91), (92), (93), we get after some lengthy calculations

$$C^{A\,B\,D}_{i\;j\;k//r}{}^{E} = B_{(1)\,r}{}^{E}\, M^{A}_{i}\, M^{B}_{j}\, M^{D}_{k} + B_{(2)\,r}{}^{E}\, N^{A}_{(1)I}\, N^{B}_{(1)j}\, N^{D}_{(1)k} + B_{(3)\,r}{}^{E}\, N^{A}_{(2)I}\, N^{B}_{(2)j}\, N^{D}_{(2)k} - \sum(\,^{A\,B\,D}_{i\;j\;k})\, L^{A}_{i}\, L^{-1}[C_{(1)}\, M^{E}_{r}\, M^{B}_{j}\, M^{D}_{k} + C_{(2)}\, N^{E}_{(1)r}\, N^{B}_{(1)j}\, N^{D}_{(1)k} + C_{(3)}\, N^{E}_{(2)r}\, N^{B}_{(2)j}\, N^{D}_{(2)k} + C_{(4)}\{M^{E}_{r}(M^{B}_{j}\, N^{D}_{(1)k} + M^{D}_{k}\, N^{B}_{(1)j}) + N^{E}_{(1)r}\, M^{B}_{j}\, M^{D}_{k}\} + C_{(5)}\{M^{E}_{r}(M^{B}_{j}\, N^{D}_{(2)k} + M^{D}_{k}\, N^{B}_{(2)j}) + N^{E}_{(2)r}\, M^{B}_{j}\, M^{D}_{k}\} + C_{(6)}\{N^{E}_{(1)r}(N^{B}_{(1)j}\, M^{D}_{k} + N^{D}_{(1)k}\, M^{B}_{j}) + M^{E}_{r}\, N^{B}_{(1)j}\, N^{D}_{(1)k}\} + C_{(7)}\{N^{E}_{(1)r}(N^{B}_{(1)j}\, N^{D}_{(2)k} + N^{B}_{(2)j}\, N^{D}_{(1)k}) + N^{E}_{(2)r}\, N^{B}_{(1)j}\, N^{D}_{(1)k}\} + C_{(8)}\{M^{E}_{r}\, N^{B}_{(2)j}\, N^{D}_{(2)k} + N^{E}_{(2)r}(M^{B}_{j}\, N^{D}_{(2)k} + M^{D}_{k}\, N^{B}_{(2)j})\} + C_{(9)}\{N^{E}_{(1)r}\, N^{B}_{(2)j}\, N^{D}_{(2)k} + N^{E}_{(2)r}(N^{B}_{(1)j}\, N^{D}_{(2)k} + N^{B}_{(2)j}\, N^{D}_{(1)k})\} + C_{(10)}\{M^{E}_{r}(N^{B}_{(1)j}\, N^{D}_{(2)k} + N^{B}_{(2)j}\, N^{D}_{(1)k}) + N^{E}_{(1)r}(M^{B}_{j}\, N^{D}_{(2)k} + M^{D}_{k}\, N^{B}_{(2)j}) + N^{E}_{(2)r}(M^{B}_{j}\, N^{D}_{(1)k} + M^{D}_{k}\, N^{B}_{(1)j})\}] + \sum(^{A\,B\,D}_{i\;j\;k})\, \{B_{(4)\,r}{}^{E}\, M^{A}_{i}\, M^{B}_{j}\, N^{D}_{(1)k} + B_{(5)\,r}{}^{E}\, M^{A}_{i}\, M^{B}_{j}\, N^{D}_{(2)k} + B_{(6)\,r}{}^{E}\, M^{A}_{i}\, N^{B}_{(1)j}\, N^{D}_{(1)k} + B_{(7)\,r}{}^{E}\, N^{A}_{(1)I}\, N^{B}_{(1)j}\, N^{D}_{(2)k} + B_{(8)\,r}{}^{E}\, M^{A}_{i}\, N^{B}_{(2)j}\, N^{D}_{(2)k} + B_{(9)\,r}{}^{E}\, N^{A}_{(1)I}\, N^{B}_{(2)j}\, N^{D}_{(2)k} + B_{(10)\,r}{}^{E}\, M^{A}_{i}(N^{B}_{(1)j}\, N^{D}_{(2)k} + N^{B}_{(2)j}\, N^{D}_{(1)k})\} \tag{95}$$

where we have taken

$$B_{(1)\,r}{}^{E} = C_{(1)//r}{}^{E} - 3\, L^{-1}(C_{(4)}\, U^{E}_{r} - C_{(5)}\, V^{E}_{r}), \tag{96}$$

$$B_{(2)\,r}{}^{E} = C_{(2)//r}{}^{E} + 3\, L^{-1}(C_{(6)}\, U^{E}_{r} - C_{(7)}\, W^{E}_{r}), \tag{97}$$

$$B_{(3)\,r}{}^{E} = C_{(3)//r}{}^{E} + 3\, L^{-1}(C_{(8)}\, V^{E}_{r} - C_{(9)}\, W^{E}_{r}), \tag{98}$$

$$B_{(4)\,r}{}^{E} = C_{(4)//r}{}^{E} + L^{-1}\{(C_{(1)} - 2\, C_{(6)})\, U^{E}_{r} + C_{(5)}\, W^{E}_{r} - C_{(10)}\, V^{E}_{r}\}, \tag{99}$$

$$B_{(5)\,r}{}^{E} = C_{(5)//r}{}^{E} + L^{-1}\{C_{(1)} - 2\, C_{(8)})V^{E}_{r} + C_{(4)}\, W^{E}_{r} - 2\, C_{(10)}\, U^{E}_{r}\}, \tag{100}$$

$$B_{(6)\,r}{}^{E} = C_{(6)//r}{}^{E} - L^{-1}\{(C_{(2)} - 2\, C_{(4)})\, U^{E}_{r} + C_{(7)}\, V^{E}_{r} + C_{(10)}\, W^{E}_{r}\}, \tag{101}$$

$$B_{(7)\,r}{}^{E} = C_{(7)//r}{}^{E} + L^{-1}\{(C_{(2)} - 2\, C_{(9)})\, W^{E}_{r} + C_{(6)}\, V^{E}_{r} + 2\, C_{(10)}\, U^{E}_{r}\}, \tag{102}$$

$$B_{(8)\,r}{}^{E} = C_{(8)//r}{}^{E} - L^{-1}\{(C_{(3)} - 2\, C_{(5)})\, V^{E}_{r} + C_{(9)}\, U^{E}_{r} - 2\, C_{(10)}\, W^{E}_{r}\}, \tag{103}$$

$$B_{(9)}\, r^{E} = C_{(9)//r}{}^{E} - L^{-1}\{C_{(3)} - 2\, C_{(7)})W^{E}_{r} - C_{(8)}\, U^{E}_{r} - 2\, C_{(10)}\, V^{E}_{r}\}, \tag{104}$$

$$B_{(10)\,r}{}^{E} = C_{(10)//r}{}^{E} + L^{-1}\{(C_{(4)} - C_{(9)})\, V^{E}_{r} + (C_{(5)} - C_{(7)})\, U^{E}_{r} + (C_{(6)} - C_{(8)})\, W^{E}_{r}\} \tag{105}$$

Hence:

Theorem 9.1: In an areal space X_4 parameterised by C_2, the covariant derivative expressed as in (18) of the tensor field $C^{A\,B\,D}_{i\;j\;k}$ is given by equation (95).

Differentiating equation $X^{i}_{A}\, C^{A\,B\,D}_{i\;j\;k} = 2\,\varphi\, h^{B\,D}_{j\;k}$, we can obtain

$$X^i_{A//r}{}^E C^{A\,B\,D}_{i\,j\,k} + X^i_A C^{A\,B\,D}_{i\,j\,k//r}{}^E = 2(\varphi_{//r}{}^E h^{B\,D}_{j\,k} + \varphi h^{B\,D}_{j\,k//r}{}^E) \tag{106}$$

Using equations (38) and (93), we get

$$X^i_{A//r}{}^E C^{A\,B\,D}_{i\,j\,k} = [C_{(1)} M^B_j M^D_k + C_{(4)}(M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j}) + C_{(5)}(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j}) + C_{(6)} N^B_{(1)j} N^D_{(1)k} + C_{(8)} N^B_{(2)j} N^D_{(2)k} + C_{(10)}(N^B_{(1)j} N^D_{(2)k} + N^D_{(1)k} N^B_{(2)j})] [\beta_{//r}{}^E + L^{-1}(\alpha/2\, M^E_r - \gamma\, U^E_r - \Theta\, V^E_r)] + [C_{(2)} N^B_{(1)j} N^D_{(1)k} + C_{(4)} M^B_j M^D_k + C_{(6)}(M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j}) + C_{(7)}(N^B_{(1)j} N^D_{(2)k} + N^B_{(2)j} N^D_{(1)k}) + C_{(9)} N^B_{(2)j} N^D_{(2)k} + C_{(10)}(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j})][\gamma_{//r}{}^E + L^{-1}(\alpha/2\, N^E_{(1)r} + \beta\, U^E_r - \Theta\, W^E_r)] + [C_{(3)} N^B_{(2)j} N^D_{(2)k} + C_{(5)} M^B_j M^D_k + C_{(7)} N^B_{(1)j} N^D_{(1)k} + C_{(8)}(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j}) + C_{(9)}(N^B_{(1)j} N^D_{(2)k} + N^B_{(2)j} N^D_{(1)k}) + C_{(10)}(M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j})][\Theta_{//r}{}^E + L^{-1}(\alpha/2\, N^E_{(2)r} + \beta\, V^E_r + \gamma\, W^E_r)] \tag{107}$$

Using equations (63) and (94), we can obtain

$$X^i_A C^{A\,B\,D}_{i\,j\,k//r}{}^E = \beta[B_{(1)\,r}{}^E M^B_j M^D_k + B_{(4)\,r}{}^E (M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j}) + B_{(5)\,r}{}^E(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j}) + B_{(6)\,r}{}^E N^B_{(1)j} N^D_{(1)k} + B_{(8)\,r}{}^E N^B_{(2)j} N^D_{(2)k} + B_{(10)\,r}{}^E(N^B_{(1)j} N^D_{(2)k} + N^B_{(2)j} N^D_{(1)k})] + \gamma[B_{(2)\,r}{}^E N^B_{(1)j} N^D_{(1)k} + B_{(4)\,r}{}^E M^B_j M^D_k + B_{(6)\,r}{}^E (M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j}) + B_{(7)\,r}{}^E (N^B_{(1)j} N^D_{(2)k} + N^D_{(1)k} N^B_{(2)j}) + B_{(9)\,r}{}^E N^B_{(2)j} N^D_{(2)k} + B_{(10)\,r}{}^E(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j})] + \Theta[B_{(3)\,r}{}^E N^B_{(2)j} N^D_{(2)k} + B_{(5)\,r}{}^E M^B_j M^D_k + B_{(7)\,r}{}^E N^B_{(1)j} N^D_{(1)k} + B_{(8)\,r}{}^E(M^B_j N^D_{(2)k} + M^D_k M^B_{(2)j}) + B_{(9)\,r}{}^E(N^B_{(1)j} N^D_{(2)k} + N^D_{(1)k} N^B_{(2)j}) + B_{(10)\,r}{}^E(M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j})] + L^{-1}[M^E_r\{C_{(1)}(\alpha\, M^B_j M^D_k + \beta\, L^B_j M^D_k + \beta\, L^D_k M^B_j) + C_{(4)}(\alpha\, (M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j}) + + (\gamma\, M^D_k + \beta\, N^D_{(1)k}) L^B_j + (\gamma\, M^B_j + \beta\, N^B_{(1)k}) L^D_k) + C_{(5)}(\alpha(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j}) + + (\Theta\, M^D_k + \beta\, N^D_{(2)k}) L^B_j + (\Theta\, M^B_j + \beta\, N^B_{(2)j}) L^D_k) + C_{(6)}(\alpha\, N^B_{(1)j} N^D_{(1)k} + \beta\, N^D_{(1)k} L^B_j + \gamma\, N^B_{(1)j} L^D_k) + C_{(8)}(\alpha\, N^B_{(2)j} N^D_{(2)k} + \Theta(N^D_{(2)k} L^B_j + N^B_{(2)j} L^D_k)) + C_{(10)}(\alpha(N^B_{(1)j} N^D_{(2)k} + N^B_{(2)j} N^D_{(1)k}) + (\Theta\, N^D_{(1)k} + \gamma\, N^D_{(2)k}) L^B_j + (\Theta\, N^B_{(1)j} + \gamma\, N^B_{(2)j}) L^D_k)\} + N^E_{(1)r}\{C_{(2)}(\alpha\, N^B_{(1)j} N^D_{(1)k} + \gamma(L^B_j N^D_{(1)k} + L^D_k N^B_{(1)j}) + C_{(4)}(\alpha\, M^B_j M^D_k + \beta(M^D_k L^B_j + M^B_j L^D_k) + C_{(6)}(\alpha(M^D_k N^B_{(1)j} + M^B_j N^D_{(1)k}) + (\beta\, N^B_{(1)j} + \gamma\, M^B_j) L^D_k + (\beta\, N^D_{(1)k} + \gamma\, M^D_k) L^B_j) + C_{(7)}(\alpha(N^B_{(1)j} N^D_{(2)k} + N^D_{(1)k} N^B_{(2)j}) + (\gamma\, N^B_{(2)j} + \Theta\, N^B_{(1)j}) L^D_k + (\gamma\, N^D_{(2)k} + \Theta\, N^D_{(1)k}) L^B_j) + C_{(9)}(\alpha\, N^B_{(2)j} N^D_{(2)k} + \Theta(N^B_{(2)j} L^D_k + N^D_{(2)k} L^B_j)) + C_{(10)}(\alpha(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j}) + (\beta\, N^B_{(2)j} + \Theta\, M^B_j) L^D_k + (\beta\, N^D_{(2)k} + \Theta\, M^D_k) L^D_j)\} + N^E_{(2)r}\{C_{(3)}(\alpha\, N^B_{(2)j} N^D_{(2)k} + \Theta(L^B_j N^D_{(2)k} + L^D_k N^B_{(2)j})) + C_{(5)}(\alpha\, M^B_j M^D_k + \beta(L^B_j M^D_k + L^D_k M^B_j)) + C_{(7)}(\alpha\, N^B_{(1)j} N^D_{(1)k} + \beta\, L^B_j N^D_{(1)k} + \gamma\, L^D_k N^B_{(1)j})) + C_{(8)}(\alpha(M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j}) + (\beta\, N^B_{(2)j} + \Theta\, M^B_j) L^D_k + (\beta\, N^D_{(2)k} + \Theta\, M^D_k) L^B_j + C_{(9)}(\alpha(N^B_{(1)j} N^D_{(2)k} + N^D_{(1)k} N^B_{(2)j}) + (\gamma\, N^B_{(2)j} + \Theta\, N^B_{(1)j}) L^D_k + (\gamma\, N^D_{(2)k} + \Theta\, N^D_{(1)k}) L^B_j + C_{(10)}(\alpha(M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j}) + (\beta\, N^B_{(1)j} + \gamma\, M^B_j) L^D_k + (\beta\, N^D_{(1)k} + \gamma\, M^D_k) L^B_j))\}] \tag{108}$$

Substituting from equations (107) and (108) in (106) and also the values of terms on the right- hand side we get on simplification

$$M^B_j M^D_k[\, C_{(1)}\{\beta_{//r}{}^E + L^{-1}(3\,\alpha/2\, M^E_r - \gamma\, U^E_r - \Theta\, V^E_r)\} + C_{(4)}\{\gamma_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(1)r} + \beta\, U^E_r - \Theta\, W^E_r)\} + C_{(5)}\{\Theta_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(2)r} + \beta\, V^E_r + \gamma\, W^E_r)\} + (\beta\, B_{(1)\,r}{}^E + \gamma\, B_{(4)\,r}{}^E + \Theta\, B_{(5)\,r}{}^E - 4\,\varphi_{//r}{}^E)] + N^B_{(1)j} N^D_{(1)k}[C_{(6)}\{\beta_{//r}{}^E + L^{-1}(3\,\alpha/2\, M^E_r - \gamma\, U^E_r - \Theta\, V^E_r)\} + C_{(2)}\{\gamma_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(1)r} + \beta\, U^E_r - \Theta\, W^E_r)\} + C_{(7)}\{\Theta_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(2)r} + \beta\, V^E_r + \gamma\, W^E_r)\} + (\beta\, B_{(6)\,r}{}^E + \gamma\, B_{(2)\,r}{}^E + \Theta\, B_{(7)\,r}{}^E - 4\,\varphi_{//r}{}^E)] + N^B_{(2)j} N^D_{(2)k}[C_{(8)}\{\beta_{//r}{}^E + L^{-1}(3\,\alpha/2\, M^E_r - \gamma\, U^E_r - \Theta\, V^E_r)\} + C_{(9)}\{\gamma_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(1)r} + \beta\, U^E_r - \Theta\, W^E_r)\} + C_{(3)}\{\Theta_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(2)r} + \beta\, V^E_r + \gamma\, W^E_r)\} + (\beta\, B_{(8)\,r}{}^E + \gamma\, B_{(9\,r}{}^E + \Theta\, B_{(3)\,r}{}^E - 4\,\varphi_{//r}{}^E)] + (M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j})[\, C_{(4)}\{\beta_{//r}{}^E + L^{-1}(3\,\alpha/2\, M^E_r - \gamma\, U^E_r - \Theta\, V^E_r)\} + C_{(6)}\{\gamma_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(1)r} + \beta\, U^E_r - \Theta\, W^E_r)\} + C_{(10)}\{\Theta_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(2)r} + \beta\, V^E_r + \gamma\, W^E_r)\} + (\beta\, B_{(4)\,r}{}^E + \gamma\, B_{(6)\,r}{}^E + \Theta\, B_{(10)\,r}{}^E)] + (M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j})\, C_{(5)}\{\beta_{//r}{}^E + L^{-1}(3\,\alpha/2\, M^E_r - \gamma\, U^E_r - \Theta\, V^E_r)\} + C_{(10)}\{\gamma_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(1)r} + \beta\, U^E_r - \Theta\, W^E_r)\} + C_{(8)}\{\Theta_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(2)r} + \beta\, V^E_r + \gamma\, W^E_r)\} + (\beta\, B_{(5)\,r}{}^E + \gamma\, B_{(10)\,r}{}^E + \Theta\, B_{(8)\,r}{}^E)] + (N^B_{(1)j} N^D_{(2)k} + N^B_{(2)j} N^D_{(1)k})\, C_{(10)}\{\beta_{//r}{}^E + L^{-1}(3\,\alpha/2\, M^E_r - \gamma\, U^E_r - \Theta\, V^E_r)\} + C_{(7)}\{\gamma_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(1)r} + \beta\, U^E_r - \Theta\, W^E_r)\} + C_{(9)}\{\Theta_{//r}{}^E + L^{-1}(3\,\alpha/2\, N^E_{(2)r} + \beta\, V^E_r + \gamma\, W^E_r)\} + (\beta\, B_{(10)\,r}{}^E + \gamma\, B_{(7)\,r}{}^E + \Theta\, B_{(9)\,r}{}^E)] + L^{-1}[M^E_r\{C_{(1)}(\,\beta\, L^B_j M^D_k + \beta\, L^D_k M^B_j) + C_{(4)}((\gamma\, M^D_k + \beta\, N^D_{(1)k}) L^B_j + (\gamma\, M^B_j + \beta\, N^B_{(1)k}) L^D_k) + C_{(5)}((\Theta\, M^D_k + \beta\, N^D_{(2)k}) L^B_j + (\Theta\, M^B_j + \beta\, N^B_{(2)j}) L^D_k) + C_{(6)}(\beta\, N^D_{(1)k} L^B_j + \gamma\, N^B_{(1)j} L^D_k) + C_{(8)}(\Theta\, (N^D_{(2)k} L^B_j + N^B_{(2)j} L^D_k)) + C_{(10)}((\Theta\, N^D_{(1)k} + \gamma\, N^D_{(2)k}) L^B_j + (\Theta\, N^B_{(1)j} + \gamma\, N^B_{(2)j}) L^D_k) + 2(L^B_j M^D_k$$

$- L^D_k M^B_j)\} + N^E_{(1)r}\{C_{(2)}\ \gamma(L^B_j N^D_{(1)k} + L^D_k N^B_{(1)j}) + C_{(4)}(\beta(M^D_k L^B_j + M^B_j L^D_k) + C_{(6)}((\beta N^B_{(1)j} + \gamma M^B_j) L^D_k + (\beta N^D_{(1)k} + \gamma M^D_k) L^B_j) + C_{(7)}((\gamma N^B_{(2)j} + \Theta N^B_{(1)j}) L^D_k + (\gamma N^D_{(2)k} + \Theta N^D_{(1)k}) L^B_j) + C_{(9)}\Theta(N^B_{(2)j} L^D_k + N^D_{(2)k} L^B_j) + C_{(10)}((\beta N^B_{(2)j} + \Theta M^B_j) L^D_k + (\beta N^D_{(2)k} + \Theta M^D_k) L^D_j) + 2(L^B_j N^D_{(1)k} - L^D_k N^B_{(1)j}) + N^E_{(2)r}\{C_{(3)}\ \Theta(L^B_j N^D_{(2)k} + L^D_k N^B_{(2)j}) + C_{(5)}\ \beta(L^B_j M^D_k + L^D_k M^B_j) + C_{(7)}(\beta L^B_j N^D_{(1)k} + \gamma L^D_k N^B_{(1)j}) + C_{(8)}((\beta N^B_{(2)j} + \Theta M^B_j) L^D_k + (\beta N^D_{(2)k} + \Theta M^D_k) L^B_j) + C_{(9)}((\gamma N^B_{(2)j} + \Theta N^B_{(1)j}) L^D_k + (\gamma N^D_{(2)k} + \Theta N^D_{(1)k}) L^B_j) + C_{(10)}((\beta N^B_{(1)j} + \gamma M^B_j) L^D_k + (\beta N^D_{(1)k} + \gamma M^D_k) L^B_j)) + 2(L^B_j N^D_{(2)k} - L^D_k N^B_{(2)j})\}] - W^E_r(N^B_{(1)j} N^D_{(2)k} + N^B_{(2)j} N^D_{(1)k} - 2 N^B_{(2)j} N^D_{(2)k})] = 0.$ (109)

Hence:

Theorem 9.2: In an areal space X_4 parameterised by C_2, equation (106), when expanded is expressed in the form of equation (109).

If the space X_4 is also C-reducible space, equation (109) shall be expressed as

$M^B_j M^D_k[\ C_{(1)}\{\beta_{//r}{}^E + L^{-1}(3\ \alpha/2\ M^E_r - \gamma\ U^E_r - \Theta\ V^E_r)\} + (\beta\ B_{(1)\ r}{}^E + \gamma\ B_{(4)\ r}{}^E + \Theta\ B_{(5)\ r}{}^E - 4\ \varphi_{//r}{}^E)] + N^B_{(1)j} N^D_{(1)k}[C_{(6)}\{\beta_{//r}{}^E + L^{-1}(3\ \alpha/2\ M^E_r - \gamma\ U^E_r - \Theta\ V^E_r)\} + (\beta\ B_{(6)\ r}{}^E + \gamma\ B_{(2)\ r}{}^E + \Theta\ B_{(7)\ r}{}^E - 4\ \varphi_{//r}{}^E)] + N^B_{(2)j} N^D_{(2)k}[C_{(8)}\{\beta_{//r}{}^E + L^{-1}(3\ \alpha/2\ M^E_r - \gamma\ U^E_r - \Theta\ V^E_r)\} + (\beta\ B_{(8)\ r}{}^E + \gamma\ B_{(9\ r}{}^E + \Theta\ B_{(3)\ r}{}^E - 4\ \varphi_{//r}{}^E)] + (M^B_j N^D_{(1)k} + M^D_k N^B_{(1)j})[C_{(6)}\{\gamma_{//r}{}^E + L^{-1}(3\ \alpha/2\ N^E_{(1)r} + \beta\ U^E_r - \Theta\ W^E_r)\} + (\beta\ B_{(4)\ r}{}^E + \gamma\ B_{(6)\ r}{}^E + \Theta\ B_{(10)\ r}{}^E)] + (M^B_j N^D_{(2)k} + M^D_k N^B_{(2)j})\ C_{(8)}\ \{\Theta_{//r}{}^E + L^{-1}(3\ \alpha/2\ N^E_{(2)r} + \beta\ V^E_r + \gamma\ W^E_r)\} + (\beta\ B_{(5)\ r}{}^E + \gamma\ B_{(10)\ r}{}^E + \Theta\ B_{(8)\ r}{}^E)] + L^{-1}[M^E_r\{C_{(1)}(\ \beta\ L^B_j M^D_k + \beta\ L^D_k M^B_j) + C_{(6)}(\beta\ N^D_{(1)k} L^B_j + \gamma\ N^B_{(1)j} L^D_k) + C_{(8)}(\Theta\ (N^D_{(2)k} L^B_j + N^B_{(2)j} L^D_k)) + 2(L^B_j M^D_k - L^D_k M^B_j)\} + N^E_{(1)r}\{C_{(6)}((\beta\ N^B_{(1)j} + \gamma\ M^B_j) L^D_k + (\beta\ N^D_{(1)k} + \gamma\ M^D_k) L^B_j) + 2(L^B_j N^D_{(1)k} - L^D_k N^B_{(1)j})\} + N^E_{(2)r}\{C_{(8)}((\beta\ N^B_{(2)j} + \Theta\ M^B_j) L^D_k + (\beta\ N^D_{(2)k} + \Theta\ M^D_k) L^B_j) + 2\ (L^B_j N^D_{(2)k} - L^D_k N^B_{(2)j})\} - W^E_r(N^B_{(1)j} N^D_{(2)k} + N^B_{(2)j} N^D_{(1)k} - 2\ N^B_{(2)j} N^D_{(2)k})] = 0,$ (110)

in which we can put $B_{(1)\ r}{}^E = C_{(1)//r}{}^E$, $B_{(2)\ r}{}^E = 3L^{-1}C_{(6)}\ U^E_r$, $B_{(3)\ r}{}^E = 3\ L^{-1}C_{(8)}\ V^E_r$, $B_{(4)\ r}{}^E = L^{-1}(C_{(1)} - 2\ C_{(8)})U^E_r$,

$B_{(5)\ r}{}^E = L^{-1}(C_{(1)} - 2\ C_{(8)})\ V^E_r$, $B_{(6)\ r}{}^E = C_{(6)//r}{}^E$, $B_{(7)\ r}{}^E = C_{(6)}\ V^E_r$, $B_{(8)\ r}{}^E = C_{(8)//r}{}^E$, $B_{(9)\ r}{}^E = -\ C_{(8)}\ U^E_r$, $B_{(10)\ r}{}^E = (C_{(6)} - C_{(8)}\ W^E_r$.

Hence:

Theorem 9.3: In a C-reducible areal space X_4 parameterised by C_2, if X^I_A is a concurrent vector field, equation (110) shall be satisfied.

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